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Research Contribution 229

Position Finding in the Presence of Parametric Uncertainty

Prepared by the Systems Evaluation Group

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02 022900.00

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Center for Naval Analyses	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

Position Finding in the Presence of Parametric Uncertainty

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

5. AUTHOR(S) (First name, middle initial, last name)

Peter J. Butterly

6. REPORT DATE January 1973	7a. TOTAL NO. OF PAGES 21	7b. NO. OF REFS 5

8a. CONTRACT OR GRANT NO. N00014-68-A-0091 b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) CRC 229 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) ________

10. DISTRIBUTION STATEMENT

Approved for Public Release; Distribution Unlimited

11. SUPPLEMENTARY NOTES ---------	12. SPONSORING MILITARY ACTIVITY Office of Naval Research Department of the Navy Washington, D. C. 20350

13. ABSTRACT

This is a two-part discussion, the first being a review of previously established results in a general treatment of position finding, the object of which is to establish comprehensive procedures applicable to all of the Navy's position finding problems. This review illustrates how such problems are influenced by quantities that appear as parameters in the formulation of the problem. The second part extends the treatment to the case where knowledge of these quantities is uncertain.

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6801

Unclassified

Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
position finding bearing observations mathematical analysis sequential analysis prior distribution and posterior distribution						

Center for Naval Analyses
an affiliate of the University of Rochester
1401 Wilson Boulevard Arlington, Virginia 22209 703/524-9400

(CNA) 268-73
12 Feb 1973

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CENTER FOR NAVAL ANALYSES

Systems Evaluation Group

Research Contribution 229

POSITION FINDING IN THE PRESENCE OF PARAMETRIC UNCERTAINTY

January 1973

Peter J. Butterly

This Research Contribution does not necessarily represent the opinion of the Department of the Navy.

Work conducted under contract N00014—68—A—0091

Enclosure (1) to (CNA)268-73 dated 12 February 1973

ABSTRACT

This memorandum has two parts. The first part is a review of previously established results in a general treatment of position finding, the object of which is to establish comprehensive procedures applicable to all of the Navy's position finding problems. This review illustrates how such problems are influenced by quantities that appear as parameters in the formulation of the problem. The second part extends the treatment to the case where knowledge of these quantities in uncertain.

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I. INTRODUCTION

Most position finding problems involve parameters in addition to the quantity or quantities of primary interest and to those which are observed. In formulating the statement of the problem or specifying the model that is applicable, it is often assumed that the values of these parameters are known or can be determined. This assumption is often justified as in the case of the following example. Position finding systems often require a knowledge of the positions from which the observations are made. If the observations are made from fixed sites, the position of each station may be ascertained by geodetic survey techniques with a degree of precision far beyond that applicable to the overall problem. Where these conditions prevail, the set of parameter values represented by the station coordinates may be assumed known without violating the validity of the model. If, on the other hand, the observations are made from moving platforms dependent on imprecise navigational systems for a knowledge of their own positions, the treatment of these parameters must take account of the uncertainties thus introduced.

The principal contribution of this memorandum is the derivation of procedures applicable to a wide range of problems in which parametric uncertainty is present. The treatment is based on a Bayesian approach to position finding and is an extension of that previously developed for the case in which the values of all of the parameters are known. This material is covered in references (1) through (4) but will be summarized in the following three sections.

II. POSITION FINDING - A GENERAL APPROACH

In order to include the greatest number and variety of possible problems in a single formulation, position finding is outlined in the following terms:

(a) The quantity whose value is of interest is denoted by the random variable x .

(b) In order to obtain information as to the value that x assumes, measurements are performed on a finite set of quantities $\alpha_1, \alpha_2 \ldots \alpha_n$ which are related to x by means of known relationships of the form:

$$\alpha_i = f_i(x) \qquad\qquad i = 1, 2, \ldots n$$

(c) Observations of the quantities α_i yield the quantities θ_i for $i = 1, 2, \ldots n$.

(d) A complete statistical description of the measurement errors is available.

This brief outline is merely intended to serve as a starting point. To establish the general context and achieve the desired objective of applicability to a diverse range of problems, some additional amplification is necessary.

An essential feature of the approach outlined is that the observations are made on quantities related to those of primary interest. Although this is often the case in position finding on account of the comparative inaccessibility of the entity whose position is of interest, the approach is not restricted solely to situations characterized by the physical and geometrical considerations in terms of which position finding is usually described. Therefore, although we shall confer the physical attributes of a position on this quantity and refer to the relationships as being determined by geometry, the generality implicit in our model permits many other options. For example, the quantity of interest x could be a vector with elements the components of position and velocity in a specified coordinate system and the relationships could depend on the propagation characteristics of electromagnetic radiation or sound in addition to geometrical factors.

To facilitate the discussion of the relationships linking the quantity of interest with the observed quantities we give two simple examples, both of which address a problem of position finding on a plane surface. In each case, a rectangular coordinate system is employed, the position of interest is denoted by its coordinates (x, y) and the observations are made from fixed stations sited at $(X_1, Y_1), (X_2, Y_2), \ldots (X_n, Y_n)$.

In the first example, the quantities subject to observation are the angular bearings of the position (x, y) from each station and the relationships are readily shown to have the form:

$$\begin{aligned} \alpha_i &= f_i(x, y) \\ &= \arctan \frac{x - X_i}{y - Y_i} \qquad\qquad i = 1, 2, \ldots n \end{aligned} \quad .$$

In this case the quantities α_i are angular bearings measured with respect to the direction of an increasing y coordinate and the above expressions indicate their relationships to the position of primary interest.

In the second example, a signal with identifiable time structure is assumed to originate at the point (x, y) and the observations consist of the differences in the times of arrival of this signal at each station. If all of the time differences are referenced with respect to the arrival time at one station - say station r , the required relationships, for this case, are readily shown to be:

$$\alpha_i = f_i(x, y)$$

$$= \frac{1}{c}\left[\left\{(x - X_i)^2 + (y - Y_i)^2\right\}^{\frac{1}{2}} - \left\{(x - X_r)^2 + (y - Y_r)^2\right\}^{\frac{1}{2}}\right]$$

$$i = 1, 2, \ldots n$$

$$i \neq r$$

where c is the transmission velocity of the signal.

An examination of these relationships indicates one way in which parameters enter into the problem. In the first example the relationships depend on the values of the 2n parameters $X_i, Y_i, i=1, 2, \ldots n$, and, in the second, there is, in addition, dependence on the parameter c . In general, a dependence on the arbitrary parameters $\gamma_1, \gamma_2, \ldots \gamma_\ell$ could be made explicit by writing:

$$\alpha_i = f_i(x)$$

$$= f_i^*(x, \gamma_1, \gamma_2, \ldots \gamma_\ell), \qquad i = 1, 2, \ldots n \quad .$$

Although one objective of this research contribution is to consider how to proceed in the presence of uncertainty with respect to such parameters, since we are here summarizing previously established results, we shall assume that the values of all of the parameters present are known precisely. It then follows, since the α_i's are the quantities subject to observation and not the data, that the relationships are entirely deterministic and consequently describe but part of the problem. Nevertheless, certain key features may be ascertained from the form they assume.

First, one consequence of the choice of a particular set of α_i's is indicated by classifying each relationship into the categories of identity, linearity or nonlinearity. The first of these categories corresponds to a direct measurement of the quantity of interest and its inclusion illustrates the applicability of the model to systems which rely partly or entirely in such direct measurements. The other two categories require no explanation and the implications of membership are self-evident. Secondly, the relationships may be further classified according to whether the set of α_i's so described are

insufficient, minimally sufficient or more than sufficient for the unique determination of the quantity of interest. Both the second and third of these categories correspond to an invertibility condition on the relationships and the third category alone to the inclusion of redundancy. In the first example given above, a minimum of two stations is necessary if the invertibility condition is to be satisfied, and three or more are required if redundancy is also a prerequisite. In the second example, the corresponding numbers are three stations for invertibility and four or more for redundancy, since, in the system considered, one station serves only as a reference.

In order to proceed, it will be assumed that the relationships can be expressed in the form given above and that the dependence of each of the quantities selected for observation on those of primary interest and on relevant incidental parameters is unambiguous. Where a viable problem is described by relationships which do not comply with this format, the appropriate treatment will differ in some respects. No assumptions will be made with respect to the linearity of the relationships and whereas it will generally be assumed that the invertibility condition is satisfied, it will be shown that the procedures established can be advantageously employed even in cases where it is not satisfied.

The inference procedures to be established utilize the data $\theta_1, \theta_2, \dots \theta_n$ obtained from the observations of the quantities $\alpha_1, \alpha_2, \dots \alpha_n$ and also a knowledge of the statistics of the errors present in the measurements. To include the algorithmic and tabular "look-up" procedures which might result from a calibration process in addition to formal probabilistic statements, it is convenient to formulate the uncertainty of the observations by means of a generating function which provides the numerical values of the probabilities defined by an applicable density function. For the choice of a density function to describe this uncertainty, two closely related alternatives will be considered; in each case the corresponding generating function must take into account any parameters on which this uncertainty may depend. If these parameters are represented by $\gamma_{\ell+1}, \gamma_{\ell+2}, \dots \gamma_s$ and the measurement errors by $\epsilon_1, \epsilon_2, \dots \epsilon_n$, the first alternative consists of specifying the generating function $g(\epsilon_1, \epsilon_2, \dots \epsilon_n)$ indicated by the correspondence:

$$p(\epsilon_1, \epsilon_2, \dots \epsilon_n) \sim g(\epsilon_1, \epsilon_2, \dots \epsilon_n)$$

or, in vector notation,

$$p(\underline{\epsilon}) \sim g(\underline{\epsilon}) \quad ,$$

which is provided by the known values of the parameters $\gamma_{\ell+1}, \dots \gamma_s$ and the generating function $g^*(\underline{\epsilon}, \gamma_{\ell+1}, \dots \gamma_s)$ for the density $p(\underline{\epsilon} | \gamma_{\ell+1}, \dots \gamma_s)$ which is assumed to be known. If, in addition it is assumed that the errors are additive, that is,

$$\epsilon_i = \theta_i - \alpha_i \, , \qquad i = 1, 2, \dots n$$

or, in vector notation,

$$\underline{\epsilon} = \underline{\theta} - \underline{\alpha} \quad ,$$

the following correspondence is readily justified:

$$p(\underline{\theta} \mid \underline{\alpha}) \sim g(\underline{\theta} - \underline{\alpha}) \quad .$$

This formulation of the uncertainty present in the observations corresponds to a direct calibration of the measurement equipment and constitutes the second of the alternatives referred to above. If the additive condition on the errors is not satisfied, direct calibration techniques will still yield the generating function appropriate to this density, but it will have the more general form indicated by a correspondence of the type:

$$p(\underline{\theta} \mid \underline{\alpha}) \sim g(\underline{\theta}, \underline{\alpha}) \quad .$$

In the sections which follow it will be assumed that the errors are additive and that a complete statement of the uncertainty of the observations is provided by the specification of a generating function for the joint error density:

$$p(\underline{\varepsilon} \mid \gamma_{\ell+1}, \ldots \gamma_s)$$

or, alternatively for the conditional density:

$$p(\underline{\theta} \mid \underline{\alpha}, \gamma_{\ell+1}, \ldots \gamma_s) \quad .$$

If the additive condition is not satisfied, knowledge of the joint error density is, per se, insufficient, and the second alternative would require a knowledge of the more general generating function indicated above.

As a final point, it should be appreciated that uncertainty in the observations may also depend on the value of the quantity of interest. A range-dependent error or one influenced by the anisotropy of the medium are examples of this. Also, the formulation of this additional dependence may introduce a further set of incidental parameters. If this dependence is completely specified, the methodology is again applicable, the additional steps necessary are presented in reference (4). Assuming that these steps have been incorporated, the situation with respect to uncertainty in the values of the parameters thus introduced is no different from that affecting incidental parameters in general. The procedures that will be established are therefore also applicable to this form of uncertainty.

III. PROCEDURES WHEN ALL PARAMETERS ARE KNOWN

In agreement with the preceding section, the problem is formulated by specifying the relationships:

$$\alpha_i = f_i(x) \qquad\qquad i = 1, 2, \ldots n$$

and a generating function $g(\underline{\epsilon})$ for the joint error density. It is also assumed that knowledge of x existing before the observations are made can be formulated by means of a discrete density function in agreement with the rationale introduced in references (1) through (3). This may be briefly stated as follows. There is some degree of precision that is adequate for the treatment of the problem. This implies that in place of the infinitely many values that x can assume, we need consider only a finite number of uncertainty regions, the dimensions of which are chosen in accordance with overall precision requirements. Since the total probability associated with any such region can be considered as concentrated at any one point within the region, prior knowledge may be taken into account by assigning probabilities to each member of a finite point set resulting from this quantization process. The dimensionality of the quantity x determines the number of subscripts necessary for proper indexing. For a one-dimensional problem, which, for the present we shall assume, the prior density function has the form:

$$\sigma(x) = \sigma_j \, \delta(x - x_j) \qquad\qquad j = 1, 2, \ldots N$$

whereas, for the two examples of position finding on a surface given in the preceding section, the form that is appropriate is:

$$\sigma(x, y) = \sigma(x - x_j \quad \delta(x - x_j, \; y - y_k) \qquad \begin{array}{l} j = -J, \ldots -1, 0, 1, \ldots J \\ k = -K, \ldots -1, 0, 1, \ldots K \end{array}$$

where $\delta(x - x_j)$ and $\delta(x - x_j, \; y - y_k)$ denote generalized univariate and bivariate functions usually referred to as Delta functions.

Given that prior knowledge may be formulated in this way, our principal objective is to establish a procedure for computing the posterior density $p(x|\underline{\theta})$ corresponding to an arbitrarily assigned prior density $\sigma(x)$ for any problem conforming with the model adopted. This density indicates how the prior knowledge is modified by the observations and therefore provides a statement of all of the information available for subsequent action. The transformation from a prior to a posterior density is, of course, achieved by means of Bayes' Rule, and, as a consequence of the argument that a discrete formulation of prior knowledge is adequate, a convenient form of this rule is given by:

$$p(x|\underline{\theta}) = q_j \, \delta(x - x_j),$$

where

$$q_j = \frac{p_j \sigma_j}{\sum_{j=1}^{N} p_j \sigma_j}$$

$$p_j = p(\underline{\theta} | x_j) \qquad\qquad j = 1, 2, \ldots N$$

and $\underline{\theta}$ is the data provided by the observations. This formulation assumes a one-dimensional problem. In the examples quoted in the preceding section the appropriate form is:

$$p(x, y \,|\, \underline{\theta}) = q_{jk} \,\delta(x - x_j, y - y_k)$$

where

$$q_{jk} = \frac{p_{jk} \sigma_{jk}}{\sum_{j=-J}^{J} \sum_{k=-K}^{K} p_{jk} \sigma_{jk}}$$

and

$$p_{jk} = p(\underline{\theta} \,| x_j, y_k) \qquad\qquad \begin{array}{l} j = -J, \ldots -1, 0, 1, \ldots J \\ k = -K, \ldots -1, 0, 1, \ldots K \end{array}$$

The extension to higher orders is obvious.

It is clear that the attributes of the position finding system, namely the relationships $f_i(x)$, $i=1, 2, \ldots n$ and the joint error density $p(\underline{\varepsilon})$ enter into Bayes' Rule via the values of the conditional density $p(\underline{\theta}|x)$ for the observed data and for $x=x_j, j=1, 2, \ldots N$. To complete the treatment, all that is therefore necessary is to indicate how this density is determined. In the preceding section it was shown that, starting with the generating function $g(\underline{\varepsilon})$ for the joint error density, and the assumption that the errors are additive we can readily establish the correspondence:

$$p(\underline{\theta} \,|\, \underline{\alpha}) \sim g(\underline{\theta} - \underline{\alpha})$$

However, if the set of relationships

$$\alpha_i = f_i(x) \qquad\qquad i = 1, 2, \ldots n$$

satisfy the invertibility condition of the previous section, then knowledge of the quantities α_i for all i is equivalent to a knowledge of x . This follows from the definition of invertibility and the assumption that all of the parameters that influence the relationships are known. Compliance with the invertibility condition therefore permits us to write, for the additive error case:

$$p(\underline{\theta}|\underline{\alpha}) \sim g(\underline{\theta} - \underline{\alpha})$$

$$= p(\underline{\theta} \mid x) \sim g\{\theta_1 - f_1(x),\ \theta_2 - f_2(x), \ldots \theta_n - f_n(x)\}$$

and $$p_j = p(\underline{\theta}|x_j) \sim g\{\theta_1 - f_1(x_j),\ \theta_2 - f_2(x_j), \ldots \theta_n - f_n(x_j)\}$$

where, in the final statement $\underline{\theta}$ consists of the observed data. This provides a generating function for the computation of the quantities $p_j, j=1, 2, \ldots N$, for insertion into the discrete form of Bayes' Rule from which the numerical values of the required posterior density may be ascertained. For nonadditive errors, the development is similar except that the initial correspondence is of the form:

$$p(\underline{\theta}|\underline{\alpha}) \sim g(\underline{\theta}, \underline{\alpha})$$

and results in a generating function for the quantities p_j which has the form:

$$g\{\theta_1, \theta_2, \ldots \theta_n, f_1(x_j), f_2(x_j), \ldots f_n(x_j)\} \qquad j=1, 2, \ldots N.$$

The entire procedure may be simply stated as follows:

(a) Replace ϵ_i by $\theta_i - f_i(x)$ for all i in the generating function for the joint error density.

(b) Replace the quantities $\theta_1, \theta_2, \ldots \theta_n$ by their observed values.

(c) Evaluate for each point x_j, $j=1, 2, \ldots N$.

(d) Replace the quantities p_j in the discrete form of Bayes' Rule by the numbers so obtained.

This provides the required posterior probabilities $q_j, j=1, 2, \ldots N$, for the case where the joint error density is available. The slight initial modification necessary where direct calibration is employed is obvious.

It should be noted however that although the development of this section has been largely possible without explicit reference to the parameters that have been introduced, this is entirely due to the assumption that prior knowledge of their precise values exists. In order to assign numerical values to the quantities p_j — and hence, to the quantities q_j, — the relationships $f_i(x)$ and the error density generating function $g(\underline{\epsilon})$ must be completely specified. This is achieved by substituting the values of the parameters known to occur in the more explicit statements of the relationships and in the generating function denoted by $f_i^*(x, \gamma_1, \gamma_2, \ldots \gamma_\ell)$ and $g^*(\underline{\epsilon}, \gamma_{\ell+1}, \gamma_{\ell+2}, \ldots \gamma_s)$, respectively.

IV. SEQUENTIAL PROCESSING WHEN ALL PARAMETERS ARE KNOWN

In the foregoing treatment it has been assumed that all of the data are simultaneously available. Where this is not the case and where the observations are of the same kind and statistically independent, it can be argued that sequential procedures offer certain advantages. The extent of the data need not be known a priori. The computational algorithms applicable to a single data point will involve only a univariate density and will be essentially the same for each step of the sequential process. The inclusion of a rule for terminating the process based on some property of the most recently computed posterior density is also a possibility. These considerations justify a re-examination of the procedures that have been formulated to determine their suitability in sequential applications.

A sequential approach to position finding differs from that previously considered. Instead of a single statement of the problem in terms of all of the available data, a number of statements are provided, each pertaining to one of an ordered subset of the data - usually a single data point, - and each of the restated problems is addressed in the order thus established. In addition, a means of combining the computations performed at each state of this process is required. In the present context a decision to adopt this approach will be greatly influenced by the availability of a common algorithm for implementing the computations appropriate to each data subset. This, in turn, depends on the homogeneity of the entire system and on whether the subsystems corresponding to the data subsets act independently of each other. Given that these matters have been settled and that a common algorithm is available, the question of combining the results of the computations performed at each stage poses no problem, since the posterior density computed at any one stage of the process becomes the prior density for the succeeding stage. A problem may arise, however, with respect to the validity of the procedures that have been established when used to obtain the computation algorithms applicable to a subsystem specified in accordance with a particular subdivision and ordering of the data.

To illustrate the problem, consider the first of the two examples of position finding on a plane surface described in section II. For sequential processing based on the single data points $\theta_1, \theta_2, \ldots \theta_n$ indexed in the order they are recorded, the relationships are as previously described, namely:

$$\alpha_i = f_i(x, y)$$
$$= \arctan \frac{x - X_i}{y - Y_i}$$

for $i=1, 2, \ldots n$, but the appropriate error density function and corresponding generating function for any single choice of i — assuming independent observations — becomes:

$$p(\varepsilon_i) \sim g(\varepsilon_i)$$

from which, if the errors are additive, we obtain:

$$p(\theta_i | \alpha_i) \sim g(\theta_i - \alpha_i) \quad .$$

Knowledge of α_i for any one value of i, is not, however, equivalent to a knowledge of (x, y), so this argument can no longer be used to equate this conditional density with $p(\theta_i|x, y)$ for which a generating function is required in order to determine the values of

$$p_{jk} = p(\theta_i|x_j, y_k)$$

for insertion into Bayes' Rule. In other words, in this example, the subsystems based on a single data point do not satisfy the invertibility condition previously used to justify this important step in the derivation of an applicable procedure.

This situation is hardly surprising. It cannot be expected that each of the subsystems corresponding to an arbitrary subdivision of the data will retain the characteristics of the corresponding composite system. Yet, while in general this will be conceded, there is intuitive support for acknowledging the equivalence of sequential and one-time processing in the particular example chosen, even when the invertibility condition is not satisfied, and consequently, for acknowledging the existence of a procedure applicable to a single data point. In fact, in a numerical example similar to this appearing in reference (2), the sequential procedures which would have been applicable, had the invertibility condition been satisfied at each stage, lead to the same posterior density as the single procedure which applies when all of the data are simultaneously available. There is, however, no inconsistency. The invertibility condition depends only on the relationship between the quantities selected for observation and the position of interest. Its utility in treating the uncertainty present in the observations is that it provides a convenient sufficient condition for establishing the equality of one formulation of this uncertainty with an alternative formulation required for the application of Bayes' Rule. It is clear, however, that any mechanism for obtaining the required formulation is acceptable.

To revert to the example, the probabilities defined by the density $p(\theta_i|\alpha_i)$ are given by the generating function $g(\theta_i - \alpha_i)$, but the application of Bayes' Rule requires the generation of the numbers p_{jk} derived from the conditional density $p(\theta_i|x, y)$. Since knowledge of α_i for any single value of i is not equivalent to a knowldge of x and y, we cannot invoke invertibility to claim that

$$p(\theta_i|x, y) = p(\theta_i|\alpha_i)$$

and that consequently

$$p_{jk} = p(\theta_i|x_j, y_k) \sim g\{\theta_i - f_i(x_j, y_k)\} .$$

Nevertheless, we claim that these statements are in fact valid, that the key equality can be justified by noting that, for any i, $i=1, \ldots n$, the independence assumed permits us to write:

$$p(\theta_i|\alpha_i) = p(\theta_i|\alpha_i, \alpha_2, \ldots \alpha_n) .$$

It is clear that:

$$p(\theta_i | \alpha_i, \alpha_2, \ldots \alpha_n) = p(\theta_i | x, y) \quad ,$$

and the remaining statements follow.

To conclude, conditional knowledge which is equivalent to knowledge of the quantity of interest with respect to a particular statement of uncertainty, or other applicable methods, may be used to effect the implementation of Bayes' Rule and thus provide acceptable procedures where, as is often the case in sequential applications, invertibility is not satisfied. Thus, in the first example of section II a single station capable of providing one angular bearing, and in the second example, two stations capable of providing only one measurement of time difference can be regarded as viable position finding systems. Where the conditions of the problem allow the choice of sequential methods, the greater flexibility and simpler processing which result makes this an attractive alternative to the one-time procedures of the preceding section.

V. PROCEDURES APPLICABLE IN THE PRESENCE OF PARAMETRIC UNCERTAINTY

In the previous sections it has been postulated that the parameters $\gamma_1, \gamma_2, \ldots \gamma_\ell$ enter into the formulation of the problem in relating the quantities selected for observation to the position of interest and that the additional parameters $\gamma_{\ell+1}, \gamma_{\ell+2}, \ldots \gamma_s$ are part of a statistical statement of the uncertainty present in the observations. The appearance of these parameters in the initial statement created no problem, since the exact values of all of them were known. If this assumption is relaxed and only some, or possibly none of the values are known precisely, the procedures described are clearly inadequate. In this section, therefore, the problem is reconsidered for the case in which r of the s parameters are precisely known, $0 \leq r < s$, and the behavior of the remainder is described by a joint density function for which a generating function is specified.

As previously stated, the attributes of a particular position finding system enter into Bayes' Rule via the quantities:

$$p_j = p(\underline{\theta} | x_j) \quad , \qquad\qquad j=1,2,\ldots N \quad .$$

Posterior probabilities can therefore be determined if an algorithm is available for computing these quantities from those given in the problem statement. To establish an algorithm that is applicable in this case, we proceed as follows. By assumption, some of the parameters which influence the joint error density may not be known, so a generating function capable of providing the numerical values of this density is not initially available. Instead we have:

$$p(\underline{\varepsilon} | \gamma_{\ell+1}, \ldots \gamma_s) \sim g^* (\underline{\varepsilon}, \ \gamma_{\ell+1}, \ldots \gamma_s)$$

namely, the generating function corresponding to a joint error density that is conditionally dependent on a knowledge of all the pertinent parameters.

If the errors are additive, then, from above:

$$p(\underline{\theta} | \underline{\alpha}, \ \gamma_{\ell+1}, \ldots \gamma_s) \sim g^* (\underline{\theta} - \underline{\alpha}, \ \gamma_{\ell+1}, \ldots \gamma_s)$$

and if, in addition, the relationships satisfy the invertibility condition, this may be used to establish the correspondence

$$p(\underline{\theta} | x, \gamma_1, \ldots \gamma_s)$$
$$\sim g\{\theta_1 - f_1^*(x, \gamma_1, \ldots \gamma_\ell) \ldots \theta_n - f_n^*(x, \gamma_1, \ldots \gamma_\ell), \gamma_{\ell+1}, \ldots \gamma_s\} \quad .$$

If either of these conditions is not satisfied, the modified procedures, described in sections III and IV, that are applicable when all the parameters are known, may still be used to obtain the generating function for this conditional density. This assigns a numerical probability for all the values that $\theta_1, \theta_2, \ldots \theta_n, x$, and $\gamma_1, \gamma_2, \ldots \gamma_s$ can each assume.

This generating function may be more general than required for the problem specified. Of the s parameters $\gamma_1, \gamma_2, \ldots \gamma_s$, the values of r are known. If the known values are substituted and the remaining $s-r$ parameters relabeled $\gamma_1, \gamma_2, \ldots \gamma_{s-r}$, a generating function which we shall designate:

$$g(\theta, x, \gamma_1, \gamma_2, \ldots \gamma_{s-r})$$

is obtained. This clearly corresponds to the density:

$$p(\underline{\theta} \mid x, \gamma_1, \gamma_2, \ldots \gamma_{s-r})$$

which is conditionally dependent on a knowledge of only those parameters which are not precisely known a priori. For these parameters, however, a joint density is specified, and, if this is denoted by:

$$\sigma(\gamma_1, \gamma_2, \ldots \gamma_{s-r})$$

in the revised notation, it is not too difficult to show that, for the position finding problem under consideration,

$$p(\underline{\theta} \mid x) = \int\int \ldots \int p(\underline{\theta} \mid x, \gamma_1, \ldots \gamma_{s-r})\, \sigma(\gamma_1, \ldots \gamma_{s-r}) d\gamma_1, \ldots d\gamma_{s-r}$$

and that

$$p_j = p(\underline{\theta} \mid x_j) = \int\int \ldots \int p(\underline{\theta} \mid x_j, \gamma_1, \ldots \gamma_{s-r})\, \sigma(\gamma_1, \ldots \gamma_{s-r}) d\gamma_1, \ldots d\gamma_{s-r} \quad .$$

By expressing the quantities p_j, $j=1,2,\ldots N$, in terms of other quantities that are either given by or can be ascertained from the statement of the problem, this provides a formal solution that is applicable where the density $\sigma(\gamma_1, \gamma_2, \ldots \gamma_{s-r})$ is continuous. If this density is discrete, or, alternatively, is continuous but can be replaced by an equivalent discrete density, the required numerical quantities are provided by an expression of the form:

$$p_j = p(x_j \mid \underline{\theta}) = \sum_{\gamma_1} \sum_{\gamma_2} \ldots \sum_{\gamma_{s-r}} p_{j, \gamma_1, \ldots \gamma_{s-r}}\, \sigma_{\gamma_1, \ldots \gamma_{s-r}}$$

$$j=1,2,\ldots N$$

in which the discrete indexing is denoted by subscripts. Using the values of p_j so obtained, the posterior density $p(x \mid \underline{\theta})$ is, as before, given by the expressions:

$$p(x \mid \underline{\theta}) = q_j \delta(x - x_j)$$

where

$$q_j = \frac{p_j \sigma_j}{\sum_j p_j \sigma_j} , \qquad j = 1, 2, \ldots N$$

for a one-dimensional problem, or for position finding on a surface by:

$$p(x, y \mid \underline{\theta}) = q_{jk} \, \delta(x - x_j, y - y_k)$$

where

$$q_{jk} = \frac{p_{jk} \sigma_{jk}}{\sum_j \sum_k p_{jk} \sigma_{jk}} ,$$

and, where the values of p_{jk} are given by, either

$$p_{jk} = \iint \ldots \int p(\underline{\theta} \mid x_j, y_k, \gamma_1, \ldots \gamma_{s-r}) \, \sigma(\gamma_1, \ldots \gamma_{s-r}) d\gamma, \ldots d\gamma_{s-r}$$

or,

$$p_{jk} = \sum_{\gamma_1} \sum_{\gamma_2} \cdots \sum_{\gamma_{s-r}} p_{j,k,\gamma_1 \ldots \gamma_{s-r}} \, \sigma_{\gamma_1 \ldots \gamma_{s-r}} ,$$

$$j = -J, \ldots -1, 0, 1, \ldots J$$
$$k = -K, \ldots -1, 0, 1, \ldots K$$

The extension to problems of higher dimensionality is obvious.

An extremely important case, closely related to that just considered, occurs where there is uncertainty with respect to certain of the parameters, but that with a view to reducing this uncertainty — for at least some of these parameters — observations are made. In this case, the use of the unconditional prior density $\sigma(\gamma_1, \gamma_2, \ldots \gamma_{s-r})$ is clearly inappropriate and should be replaced by the conditional density of these parameters given the observations that have been obtained. The necessary statement of prior knowledge with respect to the uncertain parameters is thus provided by the specification of a density which is posterior to the parameter-related observations. In this case, therefore, the solution of the position finding problem under consideration is dependent on a solution being obtained for a preliminary problem, namely, that of ascertaining this density function from the parameter-related observations and from the characteristics of the system used to obtain them. The situation may be described in more concrete terms with the aid of the example given in the introductory section in which the observations are made from mobile platforms dependent on imprecise navigational systems for

a knowledge of their own positions. Clearly, there is a subsidiary position finding problem to be considered in which the positions of the platforms are the quantities of primary interest. An independent treatment of this problem by means of the Bayesian procedures that have been established yields a posterior density which expresses what is known with respect to these positions subsequent to the observations obtained from the navigational systems. This density then provides the required statement of the uncertainty of the platform coordinates which appear as parameters in the main problem. Computation of the values of p_j or p_{jk} that are now appropriate, and hence, the posterior density of the position of primary interest may then proceed as we have indicated.

To conclude this section we observe that while the Bayesian approach that has been followed makes formal provision for the presence of uncertainty in the parameters which appear in the formulation of a position finding problem, the computational procedures are, in general, considerably simpler when the uncertainty is described by a discrete density function. If a continuous density is specified, computation of the quantities p_j for most practical problems will involve the use of numerical integration techniques. This may be avoided if the continuous density can be replaced by a discrete density. This step can be justified if the effect of the replacement can be shown to be negligible in relation to the overall precision resulting from the initial quantization of the problem. In fact, the entire question of relative precision should be investigated if unnecessary computation is to be avoided. However, in addition to the possibility that the degree of precision implicit in the initial statement of parametric uncertainty is unnecessary, there is also the possibility that it is unrealistic. An examination of the reasons for specifying a continuous density to account for the uncertainty is likely to reveal that this has been done for convenience and that the choice of a discrete density would be more consistent with applicable physical limitations.

VI. CONCLUSIONS

In many military engagements, a critical factor is the extent of the available knowledge of where something is. If the methods used to acquire this knowledge in different situations are examined, it is apparent that although the problems have much in common, there is little explicit recognition of this fact. On the contrary, the uniqueness of the particular solutions proposed is often stressed. Given the existence of this commonality, however, the advantages of a general approach are evident. It provides not only the conceptual framework necessary for the unambiguous description of the problems which occur, for the isolation of their essential features and for illustrating their relationships with other problems, but also procedures for solution, applicable to a range of problems.

The utility of this approach is governed by the variety of problems that can be effectively addressed. To provide the greatest possible generality, therefore, the formulation of position finding that is adopted and the procedures that are established are based on Bayesian methods. The selection of this approach over more conventional alternatives follows from arguments summarized in reference (5). While no restatement of these arguments will be attempted, we observe that the basic premises are simple and readily related to the kinds of problems which occur in position finding, that the introduction of ad hoc procedures for special situations is not necessary, and that sensible orthodox inference - and decision - procedures can often be reformulated in Bayesian terms. In addition, when used in conjunction with suitable preliminary quantization schemes, the unified treatment utilizing this approach (references (1) through (4)) yield general procedures for which simple computational algorithms have been derived.

The principal features of this unified treatment of position finding are summarized in sections II, III and IV of this research contribution preliminary to consideration of the kind of problem which arises in acquiring knowledge of where something is while uncertain of where you are. As a consequence of a general formulation being available, this problem is readily identified as an example of uncertainty with respect to a parameter, so it is the class of problems for which this general description holds that is addressed in the final section. The results of this section may be summarized as follows:

(a) A general treatment of position finding in a Bayesian formulation makes formal provision for the existence of parametric uncertainty and provides applicable procedures when this uncertainty is adequately specified.

(b) Explicit computational procedures are derived for the case in which prior knowledge of this uncertainty is provided by an appropriate unconditional density function and also for the case in which the prior knowledge is dependent on separate observations.

Position finding systems have been designed, constructed and used in large numbers in response to problems that have been inadequately researched or formulated. This continues, despite the existence of a general approach that can ensure both completeness and agreement with reality. This approach is provided by Bayesian methods.

REFERENCES

(1) Memorandum for Director SEG (SEG) 134-71, "Position Finding with Empirical Prior Knowledge," 21 Jul 1971

(2) Research Contribution 197, "Position Finding with Empirical Prior Knowledge," Center for Naval Analyses, 27 Dec 1971

(3) Butterly, P.J., "Position Finding with Empirical Prior Knowledge," IEEE Transactions on Aerospace and Electronic Systems, Vol. AES-8 No. 2, Mar 1972

(4) Memorandum for Director SEG (SEG) 542-72, "Position Finding with Geometry-Dependent Errors," 19 Apr 1972

(5) Lindley, D.V., "Bayesian Statistics, a Review," Society for Industrial and Applied Mathematics, Philadelphia, Pa. 1971

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